

B.Tech I yr II SEM

Subject- Strength of Materials

Topic- Theorem of Three Moments

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## Three Moment Theorem

### Three Moment Equation

The continuous beams are very common in the structural design and it is necessary to develop simplified force method known as three moment equation for their analysis. This equation is a relationship that exists between the moments at three points in continuous beam. The points are considered as three supports of the indeterminate beams. Consider three points on the beam marked as 1, 2 and 3 as shown in Figure 5.25(a). Let the bending moment at these points is  $M_1$ ,  $M_2$  and  $M_3$  and the corresponding vertical displacement of these points are  $\Delta_1$ ,  $\Delta_2$  and  $\Delta_3$ , respectively. Let  $L_1$  and  $L_2$  be the distance between points 1 – 2 and 2 – 3, respectively.

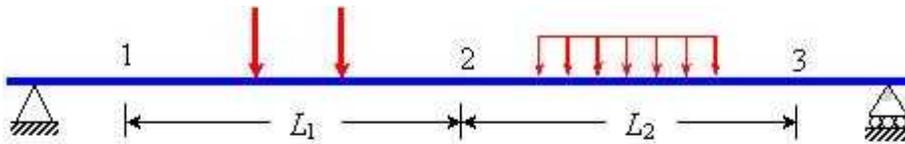


Figure 5.25(a)

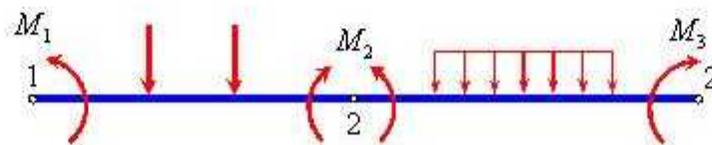


Figure 5.25(b)

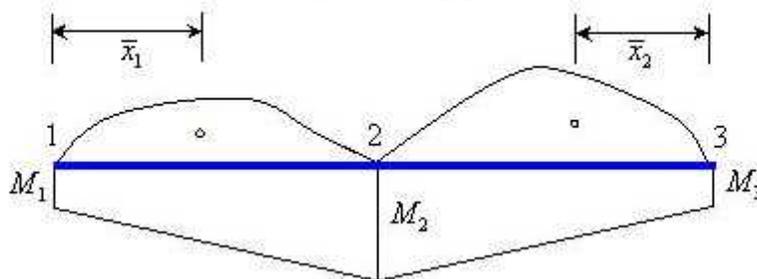


Figure 5.25(c)

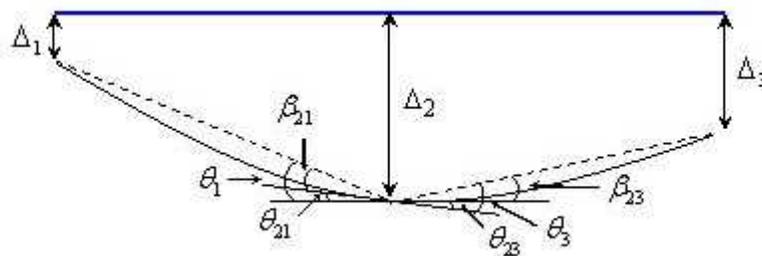


Figure 5.25(d)

The continuity of deflected shape of the beam at point 2 gives

$$\theta_{21} = \theta_{23} \quad (5.4)$$

From the Figure 5.25(d)

$$\theta_{21} = \theta_1 - \beta_{21} \quad \theta_{23} = \theta_3 - \beta_{23}$$

where

$$\theta_1 = \frac{\Delta_1 - \Delta_2}{L_1} \quad \text{and} \quad \theta_3 = \frac{\Delta_3 - \Delta_2}{L_2} \quad (5.6)$$

Using the bending moment diagrams shown in Figure 5.25(c) and the second moment area theorem,

$$\theta_{21} = \frac{1}{L_1} \times \frac{1}{EI_1} \left( \frac{M_1 L_1^2}{6} + \frac{M_2 L_1^2}{3} + A_1 \bar{x}_1 \right) \quad (5.7)$$

$$\theta_{23} = \frac{1}{L_2} \times \frac{1}{EI_2} \left( \frac{M_3 L_2^2}{6} + \frac{M_2 L_2^2}{3} + A_2 \bar{x}_2 \right) \quad (5.8)$$

where  $A_1$  and  $A_2$  are the areas of the bending moment diagram of span 1-2 and 2-3, respectively considering the applied loading acting as simply supported beams.

Substituting from Eqs. (5.7) and Eqs. (5.8) in Eqs. (5.4) and Eqs. (5.5).

$$M_1 \left( \frac{L_1}{L_1} \right) + 2M_2 \left( \frac{L_1}{L_1} + \frac{L_2}{L_2} \right) + M_3 \left( \frac{L_2}{L_2} \right) = -\frac{6A_1 \bar{x}_1}{L_1 L_1} - \frac{6A_2 \bar{x}_2}{L_2 L_2} + 6E \left[ \frac{(\Delta_2 - \Delta_1)}{L_1} + \frac{(\Delta_2 - \Delta_3)}{L_2} \right] \quad (5.9)$$

The above is known as three moment equation .

Sign Conventions

The  $M_1, M_2$  and  $M_3$  are positive for sagging moment and negative for hogging moment. Similarly, areas  $A_1, A_2$  and  $A_3$  are positive if it is sagging moment and negative for hogging moment. The displacements  $\Delta_1, \Delta_2$  and  $\Delta_3$  are positive if measured downward from the reference axis.

Example 1 Analyze the continuous beam shown in Figure 5.26(a) by the three moment equation. Draw the shear force and bending moment diagram.

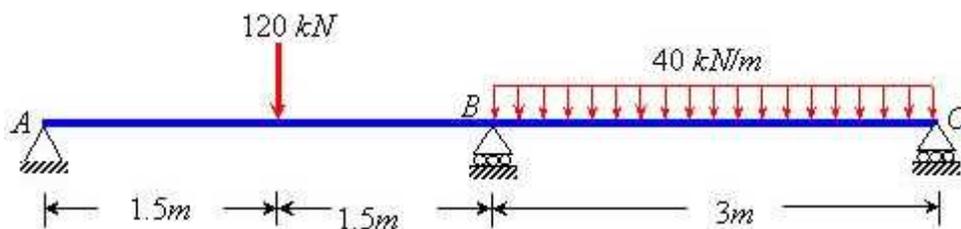
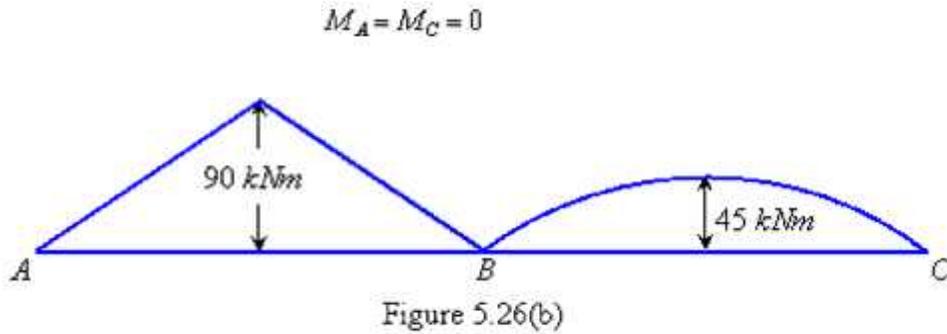


Figure 5.26(a)

Solution: The simply supported bending moment diagram on AB and AC are shown in Fig 5.26 (b). Since supports A and C are simply supported



Applying the three moment equation to span AB and BC ( $\Delta_1 = \Delta_2 = \Delta_3 = 0$ )

$$M_A \left( \frac{3}{I} \right) + 2M_B \left( \frac{3}{I} + \frac{3}{I} \right) + M_C \left( \frac{3}{I} \right) = - \frac{6 \times 1/2 \times 90 \times 3 \times 1.5}{3 \times I} - \frac{6 \times 2/3 \times 45 \times 3 \times 1.5}{3 \times I}$$

or  $M_B = -56.25 \text{ kN.m}$

The reactions at support A , B and C are given as

$$V_A = \frac{120 \times 1.5 - 56.25}{3} = 41.25 \text{ kN}$$

$$V_C = \frac{40 \times 3 \times 1.5 - 56.25}{3} = 41.25 \text{ kN}$$

$$V_B = 120 + 40 \times 3 - 41.25 - 41.25 = 157.5 \text{ kN}$$

The bending moment and shear force diagram are shown in Figures 5.26(c) and (d), respectively

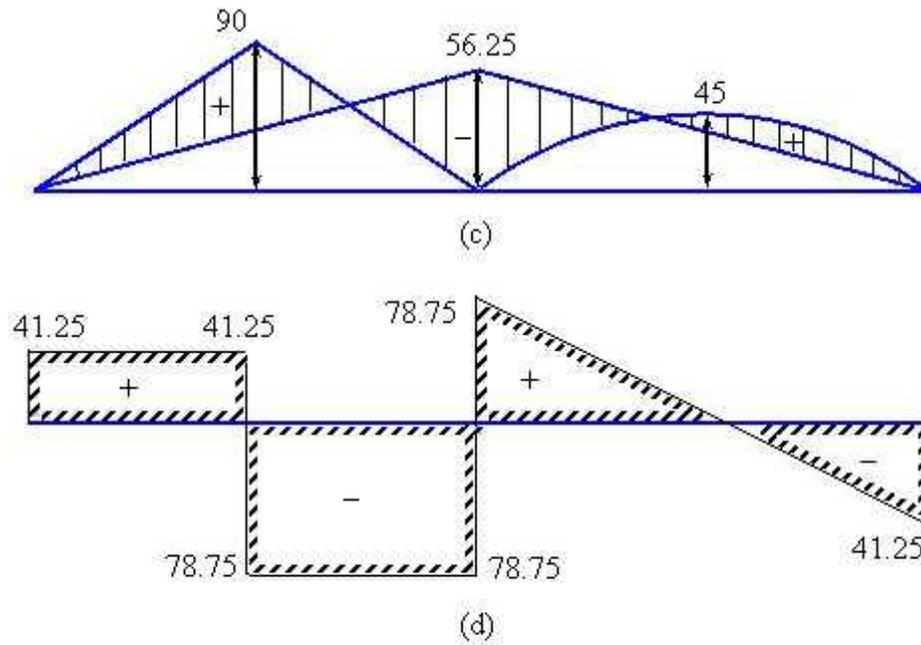


Figure 5.26(c)-(d)

Example 2 Analyze the continuous beam shown in Figure 5.27(a) by the three moment equation. Draw the shear force and bending moment diagram.

Solution: The effect of a fixed support is reproduced by adding an imaginary span  $A_0A$  as shown in Figure 5.27 (b). The moment of inertia,  $I_0$  of the imaginary span is infinity so that it will never deform and the compatibility condition at the end A, that slope should be zero, is satisfied.

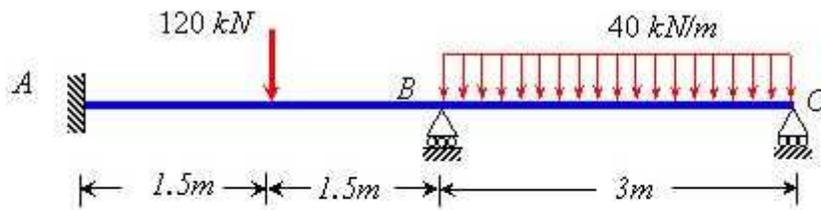


Figure 5.27(a)

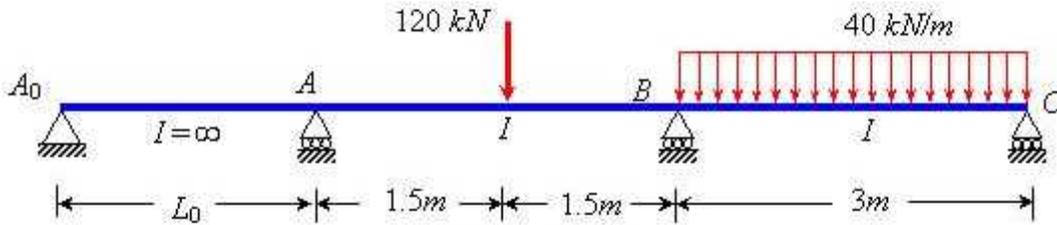


Figure 5.27(b)

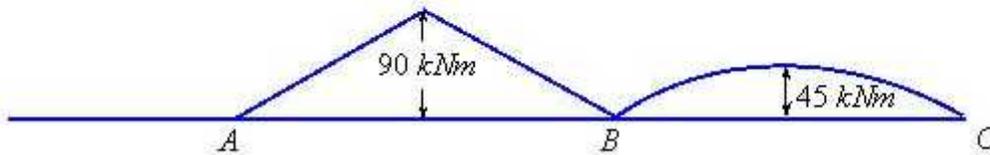


Figure 5.27(c) Simply supported moment diagram

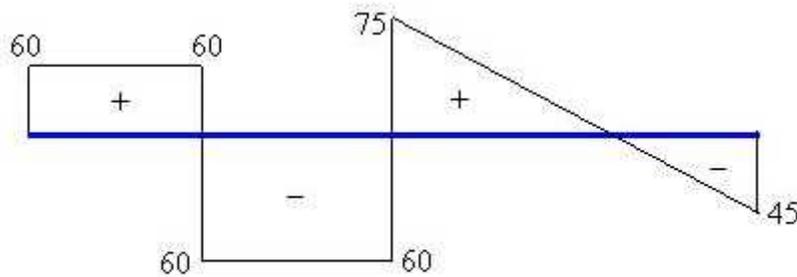


Figure 5.27(d) Shear force diagram (kN)

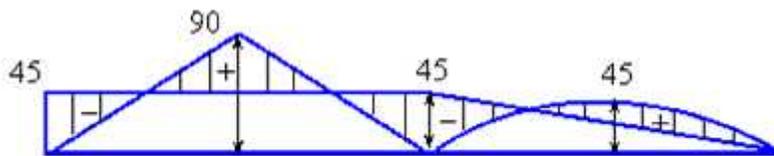


Figure 5.27(e) Bending moment diagram (kNm)

Applying three moment equation to the span  $A_0A$  and AB :

$$M_{A_0} \left( \frac{L_0}{\infty} \right) + 2M_A \left( \frac{L_0}{\infty} + \frac{3}{I} \right) + M_B \left( \frac{3}{I} \right) = - \frac{6 \times 1/2 \times 90 \times 3 \times 1.5}{3 \times I}$$

or

$$2M_A + M_B = -135 \quad (i)$$

Span AB and BC :

$$M_A \left( \frac{3}{I} \right) + 2M_B \left( \frac{3}{I} + \frac{3}{I} \right) + M_C \left( \frac{3}{I} \right) = - \frac{6 \times 1/2 \times 90 \times 3 \times 1.5}{3 \times I} - \frac{6 \times 2/3 \times 45 \times 3 \times 1.5}{3 \times I}$$

or 
$$M_A + 4M_B = - 225 \quad (ii)$$

Solving Eqs. (i) and (ii),  $M_A = - 45 \text{ kNm}$  and  $M_B = - 45 \text{ kNm}$

The shear force and bending moment diagram are shown in Figures 5.27(d) and (e), respectively.

Example 3. Analyze the continuous beam shown in Figure 5.28(a) by the three moment equation. Draw the shear force and bending moment diagram.

Solution: The simply supported moment diagram on AB, BC and CD are shown in Figure 5.28(b). Since the support A is simply supported,  $M_A = 0$ . The moment at D is  $M_D = -20 \times 2 = -40 \text{ kNm}$ .

Applying three moment equation to the span AB and BC :

$$M_A \left[ \frac{4}{I} \right] + 2M_B \left[ \frac{4}{I} + \frac{6}{3I} \right] + M_C \left[ \frac{6}{3I} \right] = - \frac{6 \times 1/2 \times 80 \times 4 \times 2}{4 \times I} - \frac{6 \times 2/3 \times 108 \times 6 \times 3}{6 \times 3I}$$

or 
$$6M_B + M_C = -456 \quad (i)$$

Span BC and CD : (  $M_D = -20 \text{ kNm}$  )

$$M_B \left[ \frac{6}{3I} \right] + 2M_C \left[ \frac{6}{3I} + \frac{6}{2I} \right] + M_D \left[ \frac{6}{2I} \right] = - \frac{6 \times 2/3 \times 108 \times 6 \times 3}{6 \times 3I} - \frac{6 \times 1/2 \times 160 \times 6 \times (6+4)/3}{6 \times 2I}$$

or 
$$M_B + 5M_C = -556 \quad (ii)$$

Solving Eqs. (i) and (ii) will give  $M_B = -59.448 \text{ kNm}$  and  $M_C = -99.310 \text{ kNm}$ .

The bending moment and shear force diagram are shown in Figures 5.28(d) and (c), respectively.

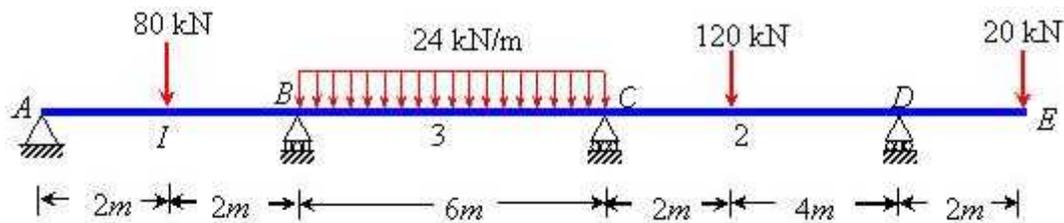


Figure 5.28(a) Given Beam

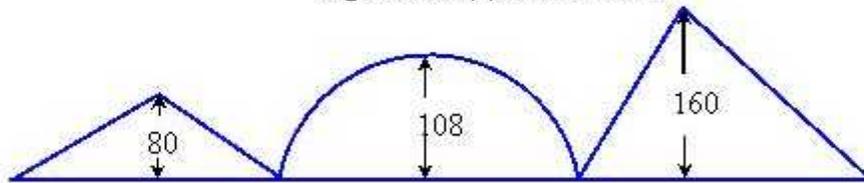


Figure 5.28(b) Simply supported Bending moment diagram (kNm)

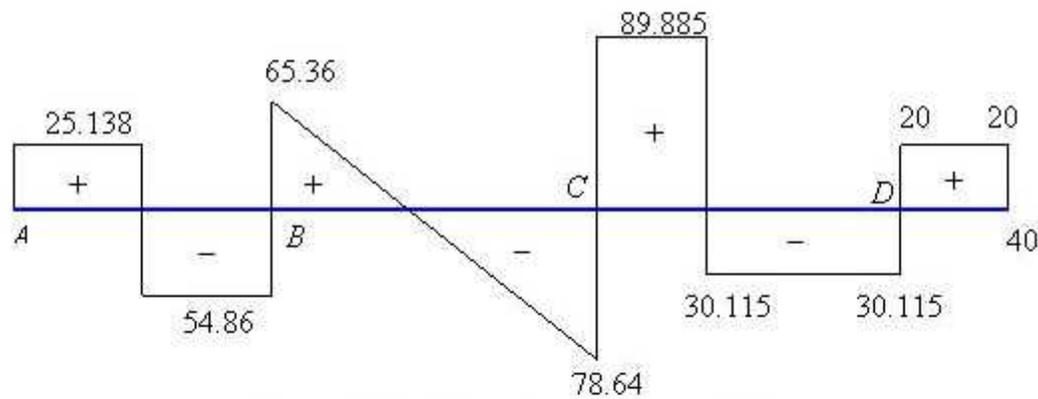


Figure 5.28(c) Shear force diagram (kN)

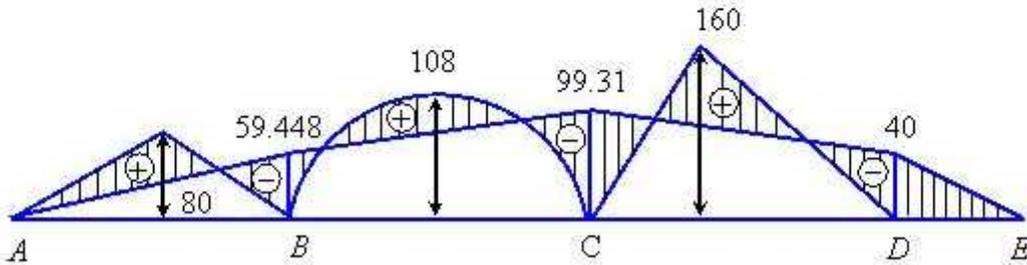


Figure 5.28(d) Bending moment diagram (kNm)

Example 4. Analyze the continuous beam show in Fig. 5.29(a) by the three moment equation method if support B sinks by an amount of 10 mm. Draw the shear force and bending moment diagram. Take flexural rigidity  $EI = 48000 \text{ kNm}^2$ .

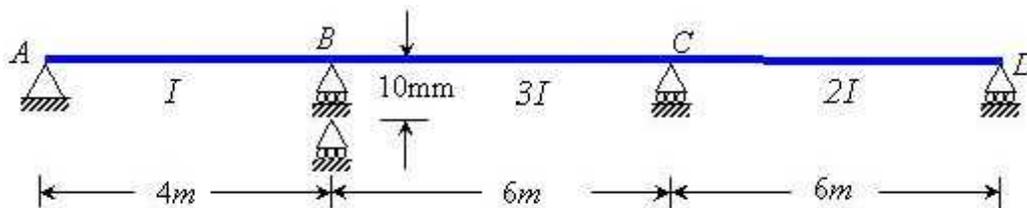


Figure 5.29(a) Given Beam

Solution: Since support A and D are simply supported,  $M_A = M_D = 0$ .

Applying the three moment equation for span AB and BC : (  $M_A = 0$  )

$$M_A \left[ \frac{4}{I} \right] + 2M_B \left[ \frac{4}{I} + \frac{6}{3I} \right] + M_C \left[ \frac{6}{3I} \right] = \frac{6 \times E \times 10 \times 10^{-3}}{4} + \frac{6E(10 \times 10^{-3})}{6}$$

or  $6M_B + M_C = 600$  (i)

Span BC and CD :

$$M_B \left[ \frac{6}{3I} \right] + 2M_C \left[ \frac{6}{3I} + \frac{6}{2I} \right] + M_D \left[ \frac{6}{2I} \right] = -\frac{6 \times E \times 10 \times 10^{-3}}{6}$$

or  $M_B + 5M_C = -240$  (ii)

Solving Eqs. (i) and (ii),  $M_B = 111.72 \text{ kNm}$  and  $M_C = -70.344 \text{ kNm}$ .

The bending moment diagram is shown in Figure 5.29(b).

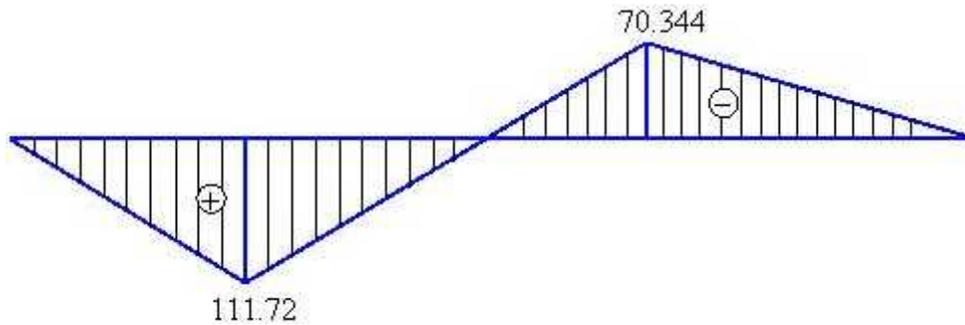


Figure 5.29(b) Bending moment diagram (kNm)